A Conjecture on the Collatz-Kakutani Path Length for the Mersenne Primes

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Abstract

We present here a new conjecture for the nature of the Mersenne prime numbers by connecting it with the Collatz-Kakutani problem. By introducing a natural *path length* on the basis of the Collatz-Kakutani tree, we conjecture that this path length of a Mersenne prime from the root of the Collatz-Kakutani tree is approximately proportional to the index of the Mersenne prime. We also discuss difference of behaviors between Mersenne numbers and Mersenne primes.

1   Introduction

Prime numbers have attracted mathematically oriented minds. They are fountains of the interesting problems which are left unresolved to the present. As well as prime numbers, natural numbers themselves sometimes show unexpected behaviors in spite of their simple appeal. In this note, we would like to present a simple property arising from the combination of two unsolved problems in number theory; the Mersenne prime numbers and the Collatz-Kakutani conjecture. Namely, we show that there exists an approximate linear relation between an index of the Mersenne prime and its Collatz-Kakutani path length, both of which are defined in the next section.

2   Mersenne primes and Collatz-Kakutani Conjecuture

Let us start with a brief descriptions of the Mersenne primes and the Collatz-Kakutani conjecture[1]. A Mersenne number is a positive integer given by

$$M_n = 2^n - 1,$$

where n is a positive integer, called "index". A Mersenne prime is a Mersenne number that is prime. It has been shown that if M_n is a Mersenne prime, n must be prime. However, the converse is not true. In fact, only forty-seven Mersenne primes have been found up to date with the largest one given when $n = 43,112,609$. This is also the largest known prime number. There are

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many fundamental questions left unresolved such as whether there are infinitely many Mersenne primes. The behaviors of Mersenne primes are irregular and unpredictable as seen in other types of prime numbers. Therefore, the recent discoveries of the Mersenne primes were achieved with the help of massively distributed computers, and the effort for finding new Mersenne primes has been made continuously [2]. The Mersenne primes are known for their relationships with the perfect numbers, and they are also applied to create pseudorandom number generators [3].

The Collatz-Kakutani conjecture is also one of the unsolved problems in number theory. This conjecture is a halting problem that the following operations will stop for an arbitrary positive number. Consider a positive integer X.

If X is odd, $3X + 1$ is the next integer.

If X is even, $X/2$ is the next integer.

If we repeat this process, it will eventually reach the halting state $X = 1$ for all positive integers. For example, if we start from $X = 7$, we have a sequence as

$$7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.$$

This process can be visualized as the Collatz-Kakutani tree as shown in Fig. 1A. The conjecture states that this tree covers all the positive integers. Even though this conjecture is unsolved, it is generally believed as true from arguments through probability theories and through computational verifications.

Let us introduce the "Collatz-Kakutani path length" $D(X)$ for a number of steps for X to reach $X = 1$ through the above operations, i.e., a number of operational steps needed to reach 1 ("the root") on the Collatz-Kakutani tree. For example, $D(7) = 16$. The relationship between X and $D(X)$ is highly irregular and no simple law for the path length is found[1]. As a result, the plot of $D(X)$ versus X produces a irregular graph as shown in Fig. 1B. While the general relationship between X and $D(X)$ is unknown, there are some trivial relations such as $D(2^n) = n$. Note that, the number 2^n is just one larger than the Mersenne number $M_n = 2^n - 1$.

3   Main Results

Our main finding to report is the fact that a path length of a Mersenne prime is approximately proportional to its index for large n, namely,

$$D(M_n) \approx 13.45n. \tag{1}$$

This is shown in Fig. 2, which we computed up to the largest known 47th Mersenne prime, $M_p(47) = M_{43112609}$. The behavior of the path length D is non–monotonic for small indices, e.g., $D(M_{89}) > D(M_{107})$. We expect, however, beyond $n = 107$, the path length increases monotonically with n for the Mersenne primes. Also, there seems no regular path to reach the root of the

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[1]In fact, it may not be defined for some X if the Collatz-Kakutani conjecture is false
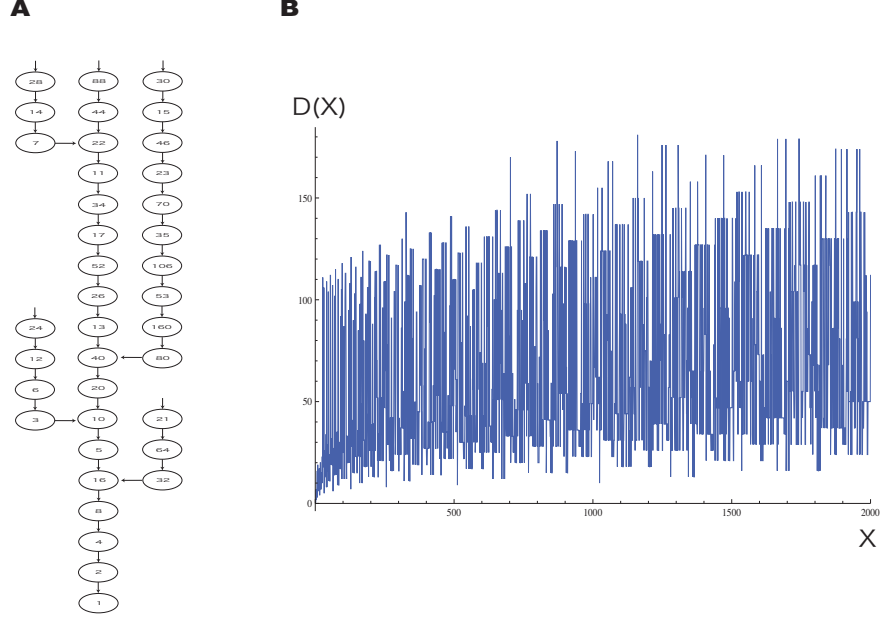
Figure 1: A. Schematic View of Collatz-Kakutani tree. B. Plot of the path length $D(X)$ on the Collatz-Kakutani tree to the root of tree $X = 1$ for integers up to X=2000.

tree, $X = 1$. For example, we have examined the path for two nearby Mersenne primes, $M_p(16) = M_{2203}$ and $M_p(17) = M_{2281}$. While they both have the path length D of approximately $30,000$, they share the last 16 steps, namely,

$22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2$, and 1.

The relations between indices of the Mersenne primes and their path lengths are summarized in Table 1.

Number	Indices	$D(X)$	$D(X)/n$
1	2	7	3.5
2	3	16	5.33333
3	5	106	21.2
4	7	46	6.57143
5	13	158	12.1538
6	17	224	13.1765
7	19	177	9.31579
8	31	450	14.5161
9	61	860	14.0984
10	89	1454	16.3371
11	107	1441	13.4673
12	127	1660	13.0709
13	521	6769	12.9923
14	607	8494	13.9934
15	1279	17094	13.3651
16	2203	29821	13.5365
17	2281	30734	13.4739
18	3217	43478	13.5151
19	4253	55906	13.1451
20	4423	60716	13.7273
21	9689	129608	13.3768
22	9941	134345	13.5142
23	11213	153505	13.6899
24	19937	265860	13.335
25	21701	293161	13.5091
26	23209	312164	13.4501
27	44497	598067	13.4406
28	86243	1158876	13.4373
29	110503	1482529	13.4162
30	132049	1771117	13.4126
31	216091	2906179	13.4489
32	756839	10197081	13.4732
33	859433	11568589	13.4607
34	1257787	16927967	13.4585
35	1398269	18807193	13.4503
36	2976221	40055567	13.4585
37	3021377	40663017	13.4584
38	6972593	93778449	13.4496
39	13466917	181209792	13.4559
40	20996011	282515044	13.4557
41	24036583	323346876	13.4523
42	25964951	349304386	13.4529
43	30402457	409093991	13.456
44	32582657	438465334	13.457
45	371566673	499902411	13.4539
46	42643801	573966881	13.4596
47	43112609	580260946	13.4592

Table 1: Path lengths of the Mersenne primes.

This linear behavior can be understood by the following heuristic arguments. Suppose N is a large random number. The standard heuristic arguments for the path length for N gives

$$D(N) \approx (3/(\ln 4/3)) \ln N. \tag{2}$$

A Mersenne number $2^n - 1$ becomes $3 \times 2^{n-1} - 1$ after two steps in Collatz-Katutani tree. Therefore, A Mersenne Number $2^n - 1$ becomes $3^n - 1$ after $2n$ steps. If we consider the number $N = 3^n - 1 \sim 3^n$ to be a large random number for large n, then the path length of the number is given by

$$D(3^n - 1) \approx (3/(\ln 4/3))n \ln 3. \tag{3}$$

Finally, we obtain the heuristic estimation of the path length of the Mersenne number to be

$$D(2^n - 1) \approx 2n + (3/(\ln 4/3))n \ln 3 \approx 13.45652n, \tag{4}$$

which is consistent with our numerical results [4, 5, 6]. This arguments implies that our finding is true not only for the Mersenne primes, but also for the Mersenne numbers. Our main conjecture, however, is that this linear relationship for the Mersenne primes are better than that for the general Mersenne numbers, or other general sequence of integers.

In order to show the difference between the Mersenne numbers in general and the Mersenne primes, the ratio of the path length to the index, $D(M_n)/n$, is plotted against n in Fig. 2B. If the distance increases linearly to the index, then $D(M_n)/n$ becomes a constant. Figure 2B shows that the Mersenne primes show better linearity than the general Mersenne numbers.
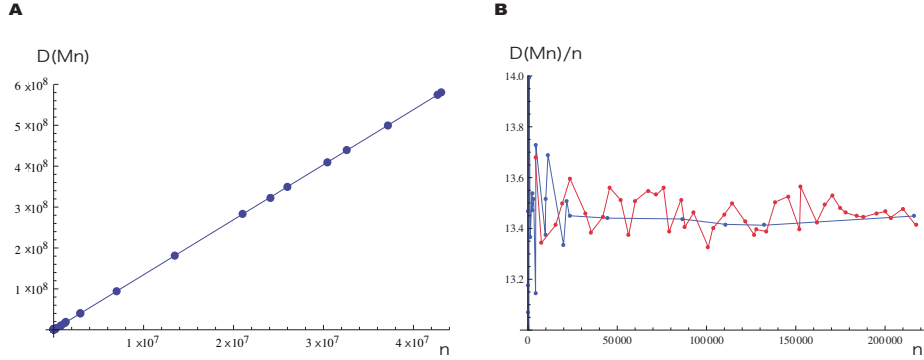

Figure 2: A. Plot of $D(M_n)$ for Mersenne primes with index n up to around $M_p(47)$. B. Plot of $D(M_n)/n$ for the Mersenne primes (Blue) and the Mersenne numbers with randomly chosen index (Red) up to around $M_p(31)$.

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To be more quantitative, we have performed the following statistical analyses. We have chosen thirteen Mersenne primes from the 26th ($M_p(26) = M_{23,209}$) to the 38th ($M_p(38) = M_{6,972,593}$), and computed the mean and the variance of $D(M_n)/n$. Let us call the prime indices of the Mersenne primes as the Mersenne prime indices. We compare the result with that from the following set of thirteen points approximately in the same interval.

A) Thirteen prime indices which is the next smallest to thirteen Mersenne prime indices. We note the Mersenne numbers associated with this set is not prime numbers themselves.

B) Thirteen indices which are near the mid-point of two successive Mersenne prime indices.

C) Thirteen indices which are twice the Mersenne prime indices. These are all even numbers. We have selected some which are close to the Mersenne prime indices.

D) Thirteen indices which is on the least squre best-fit line based on the heuristics that, given the k-th Mersenne prime $M_p(k)$, the plot of $\log_2(\log_2(M_p(k))$ versus k lies approximately on the straight line (Figure 3). We have selected some which are close to the Mersenne prime indices [7].
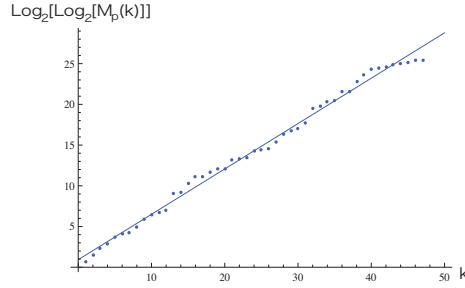

Figure 3: Plot of $\log_2(\log_2(M_p(k))$ versus k for all the known Mersenne prime up to $M_p(47)$. The straight line is obtained by the least square error fit and $\sim 0.92757 + 0.55715k$.

The results of these comparison are summarized in Table 2, showing that the path lengths of the Mersenne primes have the smaller variance than other sets. However, this difference is not enough to single out a Mersenne prime index.

In spite of the above, we look more detail into this property. We have examined the plot of $D(X = 2^n - 1)$ versus n more closely and noted that they are composed of a collection of "flat" regions with jumps like a staircase. Correspondingly, the plot of $D(X)/n$ versus n becomes collection of stripes, which interestingly appears near parallel. They are plotted in Figure 4. These plots are made for the sampled prime indices near three Mersenne prime indices, $23209, 110503, 216091$ which give 26th, 29th, and 31th Mersenne primes respectively. It is also interesting that there are occasional "overlaps" of "stair steps".

Set	Indices		$D(X)$		E[$D(X)/n$]	Var[$D(X)/n$]
Mersenne Prime	23209,	44497,	312164,	598067,	13.4473	0.0002977
	86243,	110503,	1158876,	1482529,		
	132049,	216091,	1771117,	906179,		
	756839,	859433,	10197081,	11568589,		
	1257787,	1398269,	16927967,	18807193,		
	2976221,	3021377,	40055567,	40663017,		
	6972593		93778449			
A	23227,	44501,	312182,	598071,	13.4460	0.0003194
	86249,	110527,	1158882,	1482553,		
	132059,	216103,	1771127,	2906191,		
	756853,	859447,	10197095,	11568603,		
	1257827,	1398281,	16928007,	18807205,		
	2976229,	3021407,	40055575,	40663047,		
	6972607		93778463			
B	22455,	33853,	299801,	457438,	13.4485	0.0017853
	65370,	98373,	875438,	1327329,		
	121276,	174070,	1633743,	2344640,		
	486465,	808136,	6524449,	10868120,		
	1058610,	1328028,	14246657,	17876449,		
	2187245,	2998799,	29428265,	40364153,		
	4996985		67195624			
C	22426,	43402,	299772,	584422,	13.4618	0.00132591
	46418,	88994,	627877,	1201650,		
	172486,	221006,	2320161,	2974984,		
	264098,	432182,	3556035,	5828307,		
	1513678,	1718866,	20384499,	23124964,		
	2515574,	2796538,	33827530,	37632788,		
	5952442		80085173			
D	20160,	43644,	270868,	587280,	13.4515	0.000502943
	64216,	94484,	866299,	1265873,		
	139021,	204550,	1871202,	2748585,		
	442830,	651562,	5947053,	8769774,		
	958682,	1410567,	12911249,	18991590,		
	2075452,	3053739,	27939124,	41095221,		
	4493150		60441877			

Table 2: Statistical analyses on path lengths of Mersenne primes and Mersenne numbers. E[x] denotes the arithmetic average of x and Var[x] denotes the variance of x, respectively.
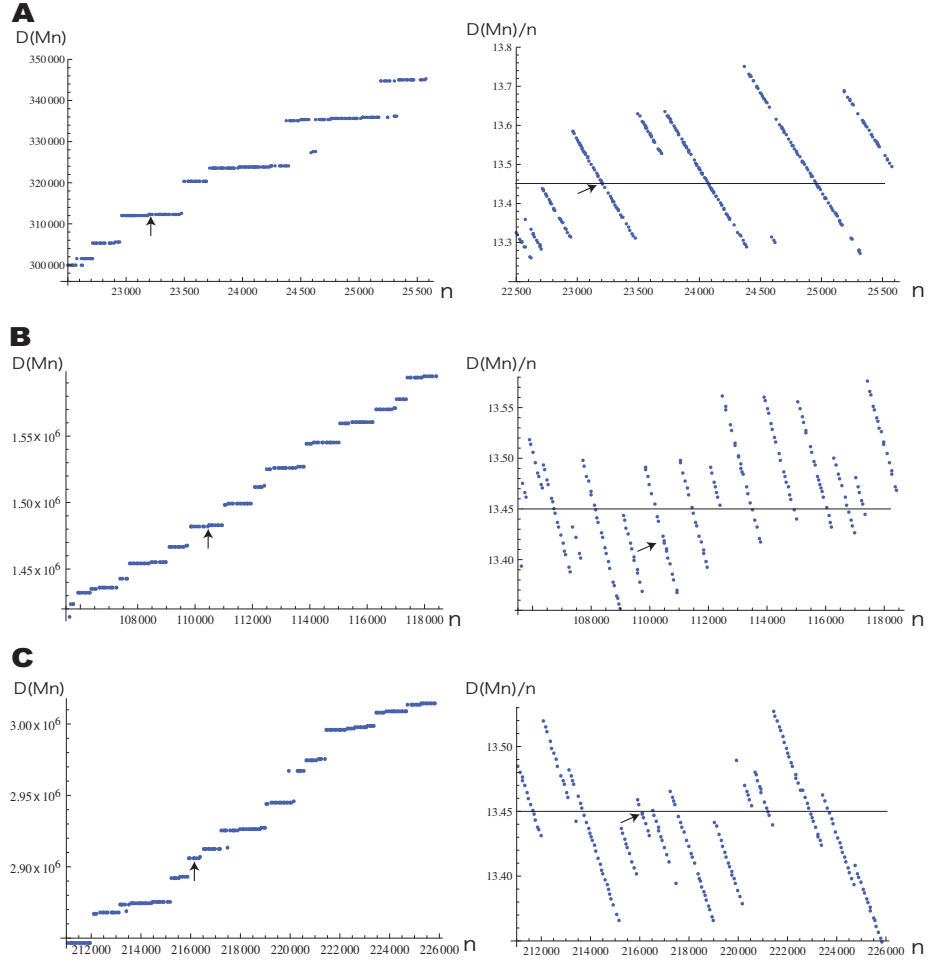
Figure 4: Plot of $D(M_n)$ (top) and $D(M_n)/n$ (bottom) versus n. The Mersenne primes are marked by arrows, and horizontal line is at $D(M_n)/n = 13.45$. A. Prime indices near $n = 23209$ ($M_p(26)$), B. Every fifth prime indices near $n = 110503$ ($M_p(29)$), C. Every fifth prime indices near $n = 216091$ ($M_p(31)$).

The nature and mechanism of these properties appeared in Figure 4 need to be studied further. We, nevertheless, conjecture that these qualitative behavior will continue to larger Mersenne numbers, and the Mersenne prime appears with its $D(X)/n \approx 13.45$. Unfortunately, again, this is clearly not sufficient to select out the Mersenne primes. If this conjecture is true, however, we can heuristically rule out points that are further away from the horizontal line of $D(X)/n \approx 13.45$ from candidates of the Mersenne primes.

4   Discussion

We have used the Collatz-Kakutani tree as a provider of "length" for natural numbers, and obtained a conjecture for the Mersenne primes. This may be yet another example where prime numbers can give rise to an emergence of unexpected order, such as Ulam Spiral [8, 9].

Though the reason behind such simple behavior is left unclear, this approach of using the Collatz-Kakutani tree can be extended to other types of set of numbers known for irregularities, possibly leading to interesting insights[10].

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